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DISCUSSION OF BUCKLING OF RIGID-JOINTED PLANE TRUSSES

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DISCUSSION

Jacob Karol,[16] M. ASCE.—This interesting paper is significant theoretically for its presentation of stiffness and carry-over factors for bars with gusset plates, and practically for the experimental results whereby the theories could be checked and a sound method of design developed. The purpose of this discussion is to show that a consistent theory of gusset plate action can be derived which checks the test results, including specimen 5 with 20% gusset plates, with an accuracy of $\pm$ 6%.

The discrepancy noted by the authors between theory and experiment, especially for specimen 5 with 20% gussets, might possibly be explained on the assumption that secondary moments reduce the primary stresses in the bars, making the actual ($k\,L$)-values in the bars for a particular load on the truss less than the theoretical values. Since no data are given in the paper comparing theoretical and measured primary stresses, it is assumed that such differences are small enough to be neglected.

The two assumptions for gusset plate rigidity given by the authors are: (1) Infinite bending rigidity over the entire length of the gusset plate and (2) hyperbolic variation of the bending rigidity from that of the member at the edge of the gusset to infinity at the center of the truss joint. The second assumption sounds correct, yet the experimental results in almost all cases fall between the theoretical results calculated from both assumptions. An obvious third assumption, suggested by the writer, is that the gusest plate is infinitely rigid over only part of its length.

A study of Figs. 4 and 5 indicates that a theoretical effective gusset length of infinite rigidity for symmetrical members given by the expression:

$$s_e = s\left[1 - \left(\frac{L - 2\,s}{L}\right)^3\right] \qquad (40)$$

fits the experimental data quite well. It is worth noting that, if the exponent of the second term in the brackets is taken as infinity, the result is full effectivity as assumed by the authors.

Further study of Figs. 4 and 5 indicates that, for the hyperbolic law assumption, the increase in stiffness is a linear function of the nominal gusset plate length. Also, the slope of the tangent at the origin to the infinite rigidity stiffness curve is twice that for the curve based on the hyperbolic law. The expressions for the stiffness, based on the two assumptions, can then be given by the expressions: For variable rigidity—

$$S = S_o + K_1\left(\frac{s}{L}\right) \qquad (41a)$$

[16] Kansas City, Mo.

and, for infinite rigidity—

$$S = S_o + 2\,K_1\left(\frac{s}{L}\right) + K_2\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots\ldots(41b)$$

in which K_1 and K_2 are constants fitting the theoretical data.

From Table 1, the theoretical buckling loads for specimen 5, based on variable rigidity, indicate a linear increase in load with an increase in gusset plate length. Consequently it has been assumed that expressions for buckling load for specimen 5 analogous to Eqs. 41*a* and 41*b* may be written: For variable rigidity—

$$W = W_o + K_3\left(\frac{s}{L}\right) \ldots\ldots\ldots\ldots\ldots\ldots(42a)$$

and, for infinite rigidity—

$$W = W_o + 2\,K_3\left(\frac{s}{L}\right) + K_4\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots\ldots(42b)$$

in which K_3 and K_4 are suitable constants. It is further assumed that Eqs. 41 and 42 apply, with proper constants, to specimens 3 and 4.

Specimens 1 and 2 will not be considered, since it is apparent that with a gusset plate at only one joint, the effect of the assumption of infinite rigidity would be to increase the buckling load only slightly more than the assumption of variable rigidity.

Several computed buckling loads in Table 1 for specimens 3, 4, and 5 were checked by the writer; hence it is assumed that all the computed loads are correct. The expressions for buckling load for gussets of infinite rigidity then are: For specimen 3—

$$W = 178 + 705{,}000\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots\ldots\ldots(43a)$$

for specimen 4—

$$W = 175 + 667\left(\frac{s}{L}\right) + 300{,}000\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots(43b)$$

and, for specimen 5—

$$W = 245 + 750\left(\frac{s}{L}\right) + 122{,}000\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots(43c)$$

Results, using Eq. 40 to determine the effective gusset length and Eqs. 43 to find the buckling loads for these lengths, are shown in Table 3. The agreement between the modified theory and actual experiment is very good in all cases.

The effect of buckling beyond the proportional limit can be considered very simply with a good degree of accuracy. Neglecting, temporarily, the increase in section of the top chord, specimen 7 is similar to specimen 5. The buckling loads, for constant E with gussets considered, would then be proportional to the buckling loads with gussets neglected. Using 437 lb as the theoretical value for

specimen 5 with 20% nominal gussets, $W_7 = 437 \times \frac{4,950}{245} = 8,830$ lb. The maximum compressive stress in the bottom chord is $\frac{P}{A} = \frac{2.5 \times 8,830}{0.391} = 56,500$ lb per sq in., and the apparent strain is $\frac{f}{E_r} = \frac{56,500}{10,500,000} = 0.00538$. From a curve of reduced modulus against stress, the stress for $\frac{f}{E_r} = 0.00538$ is 33,300 lb per sq in. and the buckling load is $\frac{33.3}{56.5} \times 8,830 = 5,200$ lb. The actual

TABLE 3.—Buckling Load of Test Specimens

Specimen No.	Gusset Plate Size (Percentage of Bar Length)		Computed Buckling Loads Based on Effective Gusset Lengths		Experimental buckling loads	Errors Based on Experimental Value (%)	
	Nominal	Effective	Infinite rigidity	Reduced modulus		Infinite rigidity	Reduced modulus
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
3	9	4	180	....	184	−2.2	....
4	9	4	203	....	216	−6.0	....
5	13	7.8	308	....	300	+2.7	....
5	20	15.7	437	....	443	−1.4	....
7	20	15.7	5,550[a]	5,200[b]	5,600	−0.9	−7.2
8	15	9.9	....	5,060[b]	5,420	....	−6.6

[a] Includes effect of reduced modulus, as described in the text. [b] Approximations based on specimen 5, as explained in the text.

buckling load will be higher because both the top chord and the web members are stressed below the proportional limit, and their stiffness, relative to that of the last two panels of the bottom chord, is increased considerably. Buckling calculations based on modified gusset lengths, infinite rigidity, and reduced modulus resulted in a buckling load of 5,550 lb.

If specimen 8 is considered similar to specimen 5, an approximate buckling load may be obtained as shown herein for specimen 7. The theoretical load for specimen 5 with 15% nominal gussets is 331 lb. Carrying through the calculations, the apparent strain is 0.00473 and the buckling load is 5,060 lb.

It is realized that the indicated percentage errors in Table 3 are correct only in so far as the assumption of similarity between (1) the shape of the curves for stiffness and (2) the shape of the curves for buckling load (expressed as functions of the gusset plate length) is valid. This assumption is based on the theoretical results for specimen 5. Its extension to the other specimens yields satisfactory agreement with experiment. The possibility of the general validity of the modified theory presented by the writer can perhaps be corroborated or refuted by the authors in their closing discussion.

Alfred S. Niles,[17] Assoc. M. ASCE.—The development of a practical procedure for taking account of the stiffening effect of gusset plates in the

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design of trusses is commendable. Prior to the presentation of this paper the theory of the stability of trusses with rigid joints had been fairly well developed; but there were few, if any, data to verify the theory, the computations were extremely tedious, and the actual stiffening effect of the joint construction was unknown.

The writer was particularly interested in the methods used for determining the stiffness and carry-over factors for beam columns of nonuniform section. The graphical method presented in 1935 by D. Williams,[18] and the writer's adaptation of that method to numerical calculation,[19] although applicable to arbitrary variation in moment of inertia, are both methods of successive approximations, and too tedious for practical application. The authors are to be congratulated for having found a direct method for obtaining these factors and presenting curves which will be easy to use in practice.

The question arises, however, as to whether the choice of a hyperbolic variation in I was due solely to the fact that it was amenable to mathematical manipulation, or whether it had any rational basis. The writer believes the former to be the case, but would like to be certain. Even though the assumption has only an empirical justification, it so simplifies the procedure of practical analysis that the paper should prove a landmark in the development of the theory of truss action.

The use of the ratio "L/j" as a measure of the intensity of loading on a member deserves some comment. This ratio has been used for some years in aeronautical work, and has proved very convenient. Since $j^2 = E\,I/P$, $\pi\,j$ is the length of a pin-ended column for which P is the critical load. It might well be termed the "Euler length" of a section for a given axial load P, just as $\pi^2\,E\,I/L^2$ is called the "Euler load" of a section for a given pin-ended length L. Because of this relationship, the distance j has a physical significance which is lacking in its reciprocal $k = \sqrt{P/E\,I}$ which is so often used in the literature of elastic stability. The writer believes this advantage outweighs the apparently greater simplicity of the formulas when k is used, but grants that the difference may be slight to all but the type-setter.

It is to be noted that the method of this paper is not well adapted to the common civil engineering practice of designing to specified allowable stresses which must not be exceeded under the maximum expected load. Therefore, it will be more enthusiastically received by the followers of J.A. Van den Broek, M. ASCE, and his predecessors in the aeronautical industry who have been using the philosophy of "limit design" for at least thirty years. To them it is the long hoped for solution to a very vexatious problem.

THOMAS C. KAVANAGH,[20] Assoc. M. ASCE.—The results of the painstaking investigation described by the authors constitute a valuable and gratifying verification of a long-accepted and used theory of stability. The aeronautical profession as a whole has made notable developments in the field of

[18] "A Successive Approximation Method of Solving the Continuous Beam Problem," by D. Williams, *Reports and Memoranda No. 1670*, Aeronautical Research Committee, London, 1935.

[19] "Airplane Structures," by Alfred S. Niles and Joseph S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1943, Vol. 2, pp. 132–140.

[20] Prof., Civ. Eng., The Pennsylvania State College, State College, Pa.

framework buckling, developments which have so unified the understanding of this problem that the civil engineering profession has only recently become aware of the pioneering work of some of its own men in this same field.

In a survey of the status of stability problems, Messrs. A. S. Niles and J. S. Newell indicated some time ago that "* * * the only type of complex structure for which there is available a reasonably satisfactory method of handling the problem of group stability is the rigid-jointed truss * * *".[21] The present experimental verification of this theory and the refinement of the theory to include gusset plate effects bring the problem nearer to final solution. Other aspects are currently under investigation by the Column Research Council of the Engineering Foundation and should eventually clarify civil engineering practice with reference to column design for end restraints.

The following itemized comments are offered with reference to the details and results of the tests:

1. The statement (under the heading "Comparison of Calculated and Measured Buckling Loads"), relative to the interaction of the two trusses, is very important and may need elaboration. If the experimental critical load of the double-truss frame is determined by the deflection of the applied load at the end of the rig, it would seem that this value would be more representative of the lower of the critical loads of the two component trusses of the frame, rather than an average of the two. It seems hardly to be expected that the workmanship of each of the two component trusses, as well as the uncontrollable effects of accidental eccentricities, and so forth, would be identical and that the two component trusses would fail at identical values. For example, did the irregularity of action noted for specimen 8 occur in both trusses simultaneously?

In other words, it would seem reasonable to expect the average critical load of the two trusses to be higher than the observed value. This possibility may be important because, although one of the major conclusions of this paper points to the loading calculated with the variable rigidity as giving closest agreement with the loading test results, the more direct approach followed in the stiffness tests cited in the paper points to a value between the variable and infinite rigidities. It is quite possible that the latter result is more nearly correct and that the direct loading test results are too low on an average basis, as indicated in the preceding paragraph.

2. Comparing the computed buckling loads for specimens 1 and 2 (Table 1), with gussets neglected versus gussets included, it appeared at first surprising that the presence of only one gusset plate in the entire truss would account for an increase of 8% in the buckling load. Closer scrutiny of the truss details, however, reveals that the "critical" (or most highly stressed) compression member is considerably stiffened at one end by this gusset plate. Engineers experienced in this type of stability analysis will recognize this phenomenon as further evidence that the truss is no stronger than its weakest part. Frequently a framework, well-proportioned throughout with the exception of one compression member, will be found to have a relatively low value of buckling load; but a slight redesign of the weak member will immediately result in a startling increase in the over-all buckling strength.

[21] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1943, Vol. II, p. 312.

3. In view of the fact that tension-member stiffnesses increase very slowly with increased loading, whereas compression-member stiffnesses decline rather rapidly with increasing loading, it is often the practice to neglect the favorable effect of axial tension. It would appear from Fig. 8 that the error from neglecting axial tension would remain at about the same relative magnitude provided the gusset plate be included in the unloaded member stiffness, whereas the error might be increased considerably if the gusset plate effect were omitted entirely.

4. No mention is made of the possible effect of slip at the riveted joints, or of the elastic behavior of such joints, which have all been considered as rigid in the analyses. Unfortunately the make-up of the joints is not indicated; but, even if this detail were included, it is probable that insufficient experimental data would be available on this specific type of connection.

It is well known that beam connections in structural framing exhibit a rotational string constant or stiffness of the order of 1×10^8 in.-lb per radian at initial low moments, and that this stiffness may decline to one twenty fifth or one fiftieth of this value at high values of the applied moment. The effect of this condition (which civil engineers refer to as "semi-rigid framing") is essentially to reduce the restraints and to lower the critical buckling load from that which would obtain had the members been rigidly attached to one another. It is possible to extend the general theory to include the effect of such elastic joints. The writer has been able to demonstrate that, using typical initial values of the spring constant of the connection as found in ordinary structural work, a reduction in the critical load of a few percent is possible due to joint elasticity.

5. The technique of direct stiffness measurement described by the authors has much to commend it as a method of verifying the theory. The writer has employed a similar principle to single axially-loaded bars to check the effective modulus experimentally, the latter modulus usually being assumed in calculations as equal to the tangent modulus.

The authors' calculated values in Figs. 4 and 5 are presumably the stiffnesses S' of the joint with the far ends elastically restrained. The authors indicate that a comparison was made with experimental values obtained from the rotation of the joint "and neighboring joints." Inasmuch as the experimental $S' = M/\theta$, the writer would like to inquire as to the need for computing rotations in adjacent joints.

As a further matter of technique, one of the principal difficulties encountered in direct stiffness tests is the necessity for keeping the applied moment low (as the authors have done) in order not to introduce secondary yielding. It would be quite possible, however, that the maximum applied moment of 400 in.-lb (assumed equally divided between the five members at the joint, and resulting in a flexural stress of $\frac{M\,c}{I} = \frac{80 \times 0.125}{\frac{1}{12} \times 0.5 \times (0.25)^3} = 15{,}360$ lb per sq in.) could give stresses in the plastic range if combined with stresses due to axial loads. The effect of such yielding would probably be evident in a curvature of the $(M\text{-}\theta)$-diagram for higher values of the moment.

The problem of the buckling of trusses is identical with that of the buckling of the individual columns of the truss, each considered as an elastically restrained

strut. As such, it should be of interest to all civil engineers, who would be alert to the economic potentialities involved in the theoretical 4-to-1 ratio of areas required for the pin-ended column versus fully-restrained column in the elastic range.[22]

In so far as the design of compression members in trusses is concerned, it is a simple matter to demonstrate with the stability theory mentioned in this paper that, for fixed loadings and the customary factors of safety employed in structural work in steel, all compression members should be designed as pin ended. Tension members reach their yield point long before the compression members are ready to buckle, and become "plastic hinges," incapable of offering restraint.

George Winter[23] and Oliver G. Julian,[24] Members, ASCE.—Among other things, this paper illustrates forcibly the interdependence of various branches of the engineering profession. It appears that the oldest branch of engineering—structural—can learn much from one of the youngest—aeronautical. This is probably because the latter is not steeped in tradition. It has been forced, by the necessity of weight saving, into the adoption of rational methods of design and analysis, largely verified by tests up to and including absolute failure.

It is the writers' understanding that the aeronautical engineer ordinarily considers the term "factor of safety" as the ratio of the loading that a structure is capable of sustaining to the imposed loading which tends to destroy the structure. "Factor of safety" so considered is a factor of overload and has real meaning. On the other hand, most structural codes and many texts are written in terms of working stress, disregarding the fact that the principle of superposition is not general. These codes leave one in an impenetrable fog as to the factor of overload, actual factor of safety, and probability of failure.

Written in terms of critical loads, this paper is refreshing, stimulating, and instructive. It is especially valuable in that it considers phenomena beyond the elastic range, investigates the influence of gusset plates mathematically, and demonstrates empirically that their effect cannot be neglected, nor can gussets reasonably be considered as infinitely rigid. It is shown that, in all cases up to and including that in which the gusset plate length at each end of the member segment is 13% of the corresponding bar length, the hyperbolic rule used gives results within a practicable degree of accuracy (5%). For longer gussets, more complicated and refined methods are in order if a close design is required. The error, in case the plates are 20% as long as the bar length, is of the order of 20% on the safe side.

The influence of riveted gusset plates on buckling loads is presently under experimental study at Cornell University, Ithaca, N. Y., as part of a project sponsored jointly by the Public Roads Administration (Department of Commerce) and the Column Research Council (Engineering Foundation). These tests differ from those reported in the paper in that they are made on full size members comprising standard structural shapes (angles and channels)

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22 "End Restraints on Truss Members," by Harold E. Wessman and Thomas C. Kavanagh, *Transactions*, ASCE, Vol. 115, 1950, p. 1135.

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24 Chf. Structural Engr., Jackson & Moreland, Cons. Engrs., Boston, Mass.

in contrast to the authors' small-scale, rectangular members. Although only half these tests are completed at this writing (1951), evidence so far indicates a significant strengthening effect of these gusset plates, which, together with the authors' evidence, points to the practical importance of accounting for this effect.

The writers believe that the inelastic range should be entered with caution. This is especially true for structures subjected to oscillating loads. Fatigue phenomena must then be given careful consideration.

The energy proof of the convergence criterion by the authors is a valuable addition to the previous work of Benjamin Wiley James[8] and Eugene E. Lundquist,[10] Assoc. M. ASCE. The paper should be studied against the background of three sources of information in addition to those cited—a paper by Mr. Lundquist[25] (1938), a paper by M. R. Horne[26] (1947), and a bulletin published by the Cornell University Engineering Experiment Station[27] (1948). The research at Cornell University[28] extends the moment distribution method, with the effect of axial forces considered, to include frames subjected to sidesway. It also gives an alternative method[29] of end restraints for determining the critical loads of frames. This method, like those of Hardy Cross,[30] Hon. M. ASCE, and R. V. Southwell[31] is essentially one of successive joint relaxation, but is distinguished from the moment distribution method presented by Messrs. James, Lundquist, and Hoff inasmuch as it results in explicit information on the actual amounts of end restraints, effective length (between hypothetical pins), and critical load for each member. Since it allows one to single out the weaker members primarily responsible for instability, it appears to have distinct advantages.

It is realized, of course, that fixed end moments, as well as the stiffness and carry-over factors, are functions of the axial forces. Equations and a chart for determining fixed end moments considering the effect of axial forces are given in the Cornell University bulletin.[32]

The statement (under the heading "The Convergence Criterion: Calculation of the Buckling Load")—

"When $\frac{L}{j}$ is greater than 2π in any one compression bar, the entire truss is unstable because the bar in question would be unstable even if its two ends were rigidly fixed."

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8 "Principal Effects of Axial Load on Moment-Distribution Analysis of Rigid Structures," by Benjamin Wiley James, *Technical Note No. 534*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1935.

10 "Tables of Stiffness and Carry-Over Factors for Structural Members Under Axial Load," by Eugene E. Lundquist and W. D. Kroll, *Technical Note No. 652*, National Advisory Committee for Aeronautics, Washington, D. C., June, 1938.

25 "Principles of Moment Distribution Applied to Stability of Structural Members," by Eugene E. Lundquist, *Proceedings*, Fifth International Cong. for Applied Mechanics, Cambridge, Mass., 1938, p. 145.

26 "A Moment Distribution Method for Rigid Frame Steel Structures Loaded Beyond the Yield Point," by M. R. Horne, Inst. of Welding, *Transactions* (*Welding Research Supplement*), London, England, Vol. 10, No. 4, August, 1947, p. 6.

27 "Buckling of Trusses and Rigid Frames," by George Winter, P. T. Hsu, Benjamin Koo, and M. H. Loh, *Bulletin No. 36*, Cornell University Engineering Experiment Station, Ithaca, N. Y. April, 1948.

28 *Ibid.*, pp. 19–21.

29 *Ibid.*, pp. 24–26.

30 "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, p. 1.

31 "Relaxation Methods in Engineering Science," by R. V. Southwell, *Oxford Engineering Science Series*, Oxford, England, Clarendon Press, 1940.

32 "Buckling of Trusses and Rigid Frames," by George Winter, P. T. Hsu, Benjamin Koo, and M. H. Loh, *Bulletin No. 36*, Cornell University Engineering Experiment Station, Ithaca, N. Y., April, 1948, pp. 13–15, and p. 56, Graph I.

—warrants further explanation. It prompts the questions: May not redundant compression bars be unstable without the entire truss being unstable? May not the compression force in a redundant compression bar exceed four times (or any other value of) the Euler load for a pin-ended member without the entire truss being unstable?

Data as to the difference in action between trusses fabricated with bolted, riveted, and welded gusset plates would be most interesting. The tests presently being conducted (1951) at Cornell indicate that the effect of bolt or rivet slip is far from negligible.

It is hoped that the studies reported by Mr. Hoff and his associates will be extended to include freestanding frames and trusses which are not braced normal to their planes. It is also hoped that the studies will be extended to include cases involving nonprismatic (and noncylindrical) members.

The writers take this opportunity to congratulate the authors on the presentation of an excellent paper. It is at once cogent, clear, and concise, and the charts present the results in such a manner as to be of real value to practicing designers.

N. J. Hoff,[33] M. ASCE, Bruno A. Boley,[34] S. V. Nardo,[35] and Sara Kaufman.[36]—The willingness to read and to check the paper as well as to comment upon it are appreciated by the writers. They are glad to answer the questions raised and wish to submit some remarks regarding the comments made. Mr. Karol suggested that some of the discrepancies between theory and experiment noted in connection with specimens having very long gusset plates (specimen 5) might be caused by the secondary stresses. The authors have not carried out an analysis of the secondary stresses in the specimens listed in the paper. It appears reasonable to assume that the secondary stresses developed in a framework with gusset plates are greater than those developed in frameworks without gusset plates in approximately the ratio of the stiffness factors of the bars having gusset plates to the stiffness factors of bars without gusset plates. This assumption is based on the customary method of calculating secondary stresses, a method by which the displacements of the joints are determined from the extensions of the bars under the primary stresses calculated for pin-jointed frameworks. The secondary stresses for the framework are then obtained by means of the Hardy Cross moment distribution method, on the assumption that these joint displacements remain unchanged. The unbalanced reactions at the joints accompanying the bending moments in the bars necessitate corrections in the primary stresses. However these corrections are insignificant for frameworks of the types shown in Fig. 3, and thus recalculation of the joint displacements on the basis of the modified primary stresses is unnecessary. Consequently the approximate procedure described must be considered as a rapidly convergent one. When the gusset plate effect is taken into account, the displacements of the equivalent pin-jointed framework are calculated exactly as before. The secondary stresses are likely to be two to three times as high as in the framework

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without gusset plates because this is the ratio of the stiffness factors of the bars having gusset plates to those of the bars without gusset plates. Consequently the corrections in the primary stress values are also doubled or tripled. With frameworks of the types investigated, these corrections will still not exceed the order of magnitude of 1%. Therefore, the secondary stress effect cannot explain the much larger discrepancy observed.

The assumption of an empirical gusset plate length that gives the same stiffness when it is considered as perfectly rigid as the actual elastic gusset plate is a useful one, because it eliminates the complex mathematical treatment which follows from the hyperbolic stiffness variation. However, the establishment of an empirical formula is advisable only if a wealth of experimental data is available. The authors did not feel that they had obtained a sufficient number of experimental points to warrant the suggestion of an empirical formula. They understand that work in this direction is now in progress at Cornell University, Ithaca, N. Y.

Even though the buckling loads calculated by Mr. Karol agree well with test results as indicated by Table 3, it is too optimistic to expect that the same agreement will be obtained with all future frameworks. His empirical formulas are based on far too scanty information. Moreover the formulas proposed for the buckling load are of little practical value because they give the buckling load of the framework having gusset plates as a function of the buckling load of a framework without gusset plates. It is no more trouble to determine the buckling load of a framework having gusset plates than to calculate the buckling load of a framework without them, as long as all the stiffness and carry-over factors can be obtained from graphs, tables, or formulas. Hence the practical engineer may just as well calculate the buckling load of a framework including the gusset plate effects.

Finally it should be mentioned that the modification of the buckling stress, used by the discusser to take into account the inelastic behavior of the material, is incorrect. If Young's modulus is replaced by the tangent modulus or the reduced modulus in the Euler formula, the buckling load or the buckling stress is obtained for the short column, This does not mean, however, that the buckling strain ϵ is given by the equation:

$$\epsilon_{cr} = \frac{\pi^2}{(L/\rho)^2} \quad \text{.......................(44)}$$

in which ϵ_{cr} is the critical strain and ρ is the radius of gyration of the cross section. The correct buckling strain is obtained by dividing the buckling stress by the corresponding secant modulus. The difference betweeen the correct critical strain and the value given incorrectly by the above equation can be easily demonstrated if some nonlinear stress-strain relationship is assumed. For instance, when the stress σ is given by the formula

$$\sigma = E\,(\epsilon - K\,\epsilon^2) \quad \text{.......................(45)}$$

the tangent modulus E_t is represented by

$$E_t = E\,(1 - 2\,K\,\epsilon) \quad \text{.......................(46)}$$

If the tangent modulus stress

$$\sigma_{cr} = \frac{\pi^2 E_t}{(L/\rho)^2} \qquad (47)$$

is considered as the correct buckling stress, substitution of the tangent modulus value leads to

$$\sigma_{cr} = \frac{\pi^2 E}{(L/\rho)^2} (1 - 2 K \epsilon_{cr}) \qquad (48)$$

On the other hand, the stress-strain relationship assumed requires that

$$\sigma_{cr} = E (\epsilon_{cr} - K \epsilon^2_{cr}) \qquad (49)$$

and thus one obtains the following quadratic equation:

$$\epsilon^2_{cr} - \epsilon_{cr} \left[\frac{1}{K} + \frac{2 \pi^2}{(L/\rho)^2} \right] + \frac{\pi^2}{K (L/\rho)^2} = 0 \qquad (50)$$

The solution of the quadratic is

$$\epsilon_{cr} = \epsilon_E + \frac{1}{2 K} - \sqrt{\epsilon^2_E + \left(\frac{1}{2 K} \right)^2} \qquad (51)$$

in which ϵ_E represents the Euler strain:

$$\epsilon_E = \frac{\pi^2}{(L/\rho)^2} \qquad (52)$$

This equation clearly shows that the critical strain ϵ_{cr} differs from the Euler strain ϵ_E, which was assumed by Mr. Karol to be the critical strain for a short column.

Mr. Niles was perfectly correct in assuming that the hyperbolic variation in the stiffness was chosen for the analysis, purely because it leads to equations that can be solved in closed form. Any other formula representing a smooth transition between the bending rigidity of the bar proper and the infinite rigidity stipulated for the mathematical intersection point of the bars would have been equally acceptable on purely physical grounds. One additional reason for the selection of the hyperbolic law was the fact that it had been used successfully at an earlier date in the calculation of the deflections of framework members.[11]

Mr. Kavanagh raises a point of great interest in connection with the interaction of the two trusses. Because the tests have shown that the critical load of a perfectly elastic truss corresponds to a true neutral equilibrium (in the sense that large deflections can take place under a load that is constant for all practical purposes), it is safe to assume that the two trusses carried equal loads at buckling when the stresses did not exceed the elastic limit. This was the case with the majority of the specimens. On the other hand the inelastic specimens

[11] "Transversely Loaded Framework Members," by N. J. Hoff, *Journal of the Royal Aeronautical Society*, Vol. XXXIX, 1935, p. 718.

buckled catastrophically. It is obvious, therefore, that the buckling of the weaker or more highly loaded truss must have thrown additional loads upon the second truss and precipitated its buckling. The experimental buckling loads of the inelastic specimens must be assumed, therefore, to be lower than the true values, as Mr. Kavanagh has pointed out. However, the writers believe that the inaccuracy resulting from this condition must be a minor one because of the care taken in distributing the loads evenly between the two trusses. In any case the conclusions reached on the basis of the fully elastic specimens need not be modified. Mr. Kavanagh was right in assuming that the irregularity noted for specimen 8 occurred in only one of the trusses.

It was not the purpose of the paper to evaluate the economy of present-day structures but rather to provide a somewhat more refined tool for the analysis of the ultimate load of a framework. It is hoped that the availability of such a tool will ultimately lead to design specifications which will permit the best use to be made of the structural materials.

The writers were very glad to learn from the discussion of Mr. Winter and Mr. Julian that the strengthening effect of the gusset plates has been observed in current tests at Cornell University also. They are also willing to admit the inaccuracy of one of their statements in the paper. The sentence (under the heading "The Convergence Criterion: Calculation of the Buckling Load")

> "When $\frac{L}{j}$ is greater than 2π in any one compression bar, the entire truss is unstable because the bar in question would be unstable if its two ends were rigidly fixed."

is correct only for statically determinate frameworks. Naturally a redundant member of a truss can be loaded beyond its buckling load without impairing the load-carrying capacity of the truss. On the other hand the convergence criterion is equally applicable to statically determinate and redundant frameworks.

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